A TOPOLOGICAL CENTRAL POINT THEOREM

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Abstract. In this paper a generalized topological central point theorem is proved for maps of a simplex to finite-dimensional metric spaces. Similar generalizations of the Tverberg theorem are considered.

1. Introduction

Let us state the discrete version of the Neumann–Rado theorem [9, 11, 5] (see also the reviews [4] and [3]):

Theorem (The discrete central point theorem). *Suppose $X \subset \mathbb{R}^d$ is a finite set with $|X| = (d+1)(r-1)+1$. Then there exists $x \in \mathbb{R}^d$ such that for any halfspace $H \ni x$*

$$|H \cap X| \geq r.$$

In this theorem a halfspace is a set $\{x \in \mathbb{R}^d : \lambda(x) \geq 0\}$ for a (possibly not homogeneous) linear function $\lambda : \mathbb{R}^d \to \mathbb{R}$. Using the Hahn–Banach theorem [12] we restate the conclusion of this theorem as follows: the point x is contained in the convex hull of any subset $F \subseteq X$ of at least $d(r-1)+1$ points.

When stated in terms of convex hulls, the central point theorem has an important and nontrivial generalization proved in [15]:

Theorem (Tverberg's theorem). *Consider a finite set $X \in \mathbb{R}^d$ with $|X| = (d+1)(r-1)+1$. Then X can be partitioned into r subsets $X_1, \dots, X_r$ so that*

$$\bigcap_{i=1}^{r} \operatorname{conv} X_i \neq \emptyset.$$

In [2, 16] a topological generalization of the Tverberg theorem was established. Instead of taking a finite point set in $\mathbb{R}^d$ and the convex hulls of its subsets, we take the continuous image of a simplex in $\mathbb{R}^d$ and the images of its faces (faces of the simplex viewed as a simplicial complex):

Theorem (The topological Tverberg theorem). *Let $m = (d+1)(r-1)$, r be a prime power, and let Δ^m be the m-dimensional simplex. Suppose $f : \Delta^m \to Y$ is a continuous map to a d-dimensional manifold Y. Then there exist r disjoint faces $F_1, \dots, F_r \subset \Delta^m$ such that*

$$\bigcap_{i=1}^{r} f(F_i) \neq \emptyset.$$

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It is still unknown whether such a theorem holds for r not equal to a prime power. But if we return to the central point theorem, we see that the following topological version holds without restrictions on r. Moreover, the target space can be any d-dimensional metric space, not necessarily a manifold. So the main result of this paper is:

Theorem 1.1. *Let $m = (d+1)(r-1)$, let Δ^m be the m-dimensional simplex, and let W be a d-dimensional metric space. Suppose $f : \Delta^m \to W$ is a continuous map. Then*

$$\bigcap_{\substack{F \subset \Delta^m \\ \dim F = d(r-1)}} f(F) \neq \emptyset,$$

where the intersection is taken over all faces of dimension $d(r-1)$.

Note that for $W = \mathbb{R}^d$ this theorem can also be deduced from the topological Tverberg theorem (see Section 4 for details). The goal of this paper is to give another proof of Theorem 1.1, valid for any d-dimensional W. In Section 5 we show that a similar generalization of the Tverberg theorem for maps into finite-dimensional spaces essentially needs larger values of m.

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2. Index of $\mathbb{Z}_2$-spaces

Let us recall some basic facts on the homological index of $\mathbb{Z}_2$-actions ($\mathbb{Z}_2$ is a group with two elements); the reader may consult the book [8] for more details. Denote $G = \mathbb{Z}_2$, if we consider $\mathbb{Z}_2$ as a transformation group. The algebra $H^*(BG; \mathbb{F}_2)$ is a polynomial ring $\mathbb{F}_2[c]$ with the one-dimensional generator c.

In this section we consider the cohomology with $\mathbb{F}_2$ coefficients, the coefficients being omitted from the notation. Define the equivariant cohomology for a space X with continuous action of G (a G-space) by

$$H^*_G(X) = H^*(X \times_G EG) = H^*((X \times EG)/G),$$

thus we have $H^*_G(\mathrm{pt}) = H^*(BG)$ for a one-point space with trivial action of G and $H^*_G(X) = H^*(X/G)$ for a free G-space. For $G = \mathbb{Z}_2$ we may take EG to be the infinite-dimensional sphere S^∞ with the antipodal action of G, and $BG = \mathbb{R}P^\infty$. For any G-space X the natural map $X \to \mathrm{pt}$ induces the natural cohomology map

$$\pi^*_X : H^*_G(\mathrm{pt}) = H^*(BG) \to H^*_G(X).$$

Definition 2.1. For a G-space X define $\operatorname{ind}_G X$ to be the maximal n such that $\pi^*_X(c^n) \neq 0 \in H^*_G(X)$.

Note that if X has a G-fixed point then the map π^*_X is necessarily injective and the index is infinite. The following property of index is obvious by definition:

Lemma 2.2. *If X is a topological disjoint union of G-invariant subspaces $X_1, \ldots, X_k$, then*

$$\operatorname{ind}_G X = \max_i \operatorname{ind}_G X_i.$$

The next property is the generalized Borsuk–Ulam theorem (see [8] for example):

Lemma 2.3. *Let $\operatorname{ind}_G X \geq n$ and let V be an n-dimensional vector space with antipodal G-action. Then for every continuous G-equivariant map $f : X \to V$*

$$f^{-1}(0) \neq \emptyset.$$

The following lemma is proved in [20], see also [6]:

Lemma 2.4. *Let X be a compact metric G-space, $\operatorname{ind}_G X \ge (d+1)k$, and let W be a d-dimensional metric space with trivial G-action. Then for every continuous G-equivariant map $f : X \to W$ there exists $x \in W$ such that*

$$\operatorname{ind}_G f^{-1}(x) \ge k.$$

In this lemma it is important to use the Čech cohomology, which is assumed in the sequel.

3. Proof of Theorem 1.1

Consider a continuous map $f : \Delta^m \to W$. Let us map the m-dimensional sphere S^m to Δ^m by the formula:

$$g(x_1, \dots, x_{m+1}) = (x_1^2, \dots, x_{m+1}^2).$$

Apply Lemma 2.4 to the composition $f \circ g$, which is possible because $g(x) = g(-x)$. We obtain a point $x \in W$ such that for $Z = (f \circ g)^{-1}(x)$ we have $\operatorname{ind}_G Z \ge r - 1$.

We are going to show that x is the required intersection point. Assume the contrary: a face $F \subseteq \Delta^m$ of dimension $d(r-1)$ does not intersect $g(Z)$. Without loss of generality, let $g^{-1}(F)$ be defined by the equations

$$x_1 = \dots = x_{r-1} = 0.$$

Note that the $r-1$ coordinates $x_1, \dots, x_{r-1}$ give a continuous G-equivariant map $h : S^m \to \mathbb{R}^{r-1}$, where G acts on $\mathbb{R}^{r-1}$ antipodally. By Lemma 2.3 the intersection $g^{-1}(F) \cap Z = h^{-1}(0) \cap Z = h|_Z^{-1}(0)$ should be nonempty. The proof is complete.

4. Remark on the case $W = \mathbb{R}^d$ of Theorem 1.1

Recall the known fact: The case $W = \mathbb{R}^d$ of Theorem 1.1 follows from the topological Tverberg theorem (only the case of prime r is needed). For the reader's convenience we present a proof here (see also [7, Section 6]).

Consider a simplicial map $\varphi : \Delta^M \to \Delta^m$, where $R = k(r-1)+1$ is a prime (for some k this is so by the Dirichlet theorem on arithmetic progressions), $M = (R-1)(d+1)+k-1$, and there are k vertices of Δ^M in the preimage of every vertex of Δ^m. For Δ^M the topological Tverberg theorem holds (since $M \ge (R-1)(d+1)$), and there exist R disjoint faces $\tilde{F}_1, \dots, \tilde{F}_R$ of Δ^M such that

$$\bigcap_{i=1}^{R} f(\varphi(\tilde{F}_i)) \ni x.$$

Consider a face $F \subseteq \Delta^m$ of dimension $d(r-1)$ and assume that $\varphi^{-1}(F)$ does not contain any $\tilde{F}_i$, then $M+1$ must be at least the number of vertices in $\varphi^{-1}(F)$ plus R, that is

$$M + 1 \ge k(r-1)d + k + R = (R-1)d + k + R = M + 2,$$

which is a contradiction. So $\varphi^{-1}(F)$ contains some $\tilde{F}_i$, and $f(F) \ni x$.

5. Tverberg type theorems for maps to finite-dimensional spaces

It is natural to ask whether the corresponding version of the Tverberg theorem holds for maps from Δ^m to a d-dimensional metric space, at least for r a prime power. In fact, the number $m = (d+1)(r-1)$ must be increased, as claimed by the following:

Theorem 5.1. *Let $m = (d+1)r - 2$. Then there exists a d-dimensional polyhedron W and a continuous map $f : \Delta^m \to W$ with the following property. For any pairwise disjoint faces $F_1, \dots, F_r \subseteq \Delta^m$ there exists i such that*

$$f(F_i) \cap f(F_j) = \emptyset$$

for all $j \neq i$.

This theorem also shows that our approach used to prove Theorem 1.1 cannot be applied to the topological Tverberg theorem. Indeed, this proof does not distinguish between $\mathbb{R}^d$ and any metric d-dimensional space, but the topological Tverberg theorem does not hold for maps to d-dimensional metric spaces.

Proof of Theorem 5.1. The construction in the proof is taken from [19]. Let Δ^m be a regular simplex in $\mathbb{R}^m$, centered at the origin. Denote by Δ^m_{d-1} its $(d-1)$-skeleton, and $W = C\Delta^m_{d-1}$ the cone (geometrically centered at the origin) on this skeleton. Define the PL-map (of the barycentric subdivision to the barycentric subdivision) $f : \Delta^m \to W$ as follows. For every face $F \subseteq \Delta^m$ of dimension $\le d-1$ its barycenter is mapped to itself, for every face $F \subseteq \Delta^m$ of dimension $\ge d$ its barycenter is mapped to the origin.

Let $F_1, \dots, F_r \subseteq \Delta^m$ be a set of r pairwise disjoint faces. For some i the dimension $\dim F_i$ is at most $d-1$ by the pigeonhole principle. For such a face we have $f(F_i) = F_i$, and

$$f(F_i) \cap f(F_j) \subseteq F_i \cap f(F_j) \subseteq \partial \Delta^m.$$

Since $f(F_j) \cap \partial\Delta^m \subseteq F_j$ we obtain

$$f(F_i) \cap f(F_j) \subseteq F_i \cap F_j = \emptyset.$$

□

The following positive result for larger m is a direct consequence of the reasoning in [18]:

Theorem 5.2. *Let $m = (d+1)r - 1$ and let r be a prime power. Suppose $f : \Delta^m \to W$ is a continuous map to a d-dimensional metric space W. Then there exist r disjoint faces $F_1, \dots, F_r \subset \Delta^m$ such that*

$$\bigcap_{i=1}^r f(F_i) \neq \emptyset.$$

Proof. Without loss of generality we may assume W to be a finite d-dimensional polyhedron. Assume the contrary and denote Δ^m by K for brevity. Then there exists a map

$$\tilde f : K^{*r}_{\Delta(2)} \to W^{*r}_{\Delta(r)}$$

from the r-fold pairwise deleted join $K^{*r}_{\Delta(2)}$ in the simplicial sense to the r-fold r-wise deleted join $W^{*r}_{\Delta(r)}$ in the topological sense (see the definitions of the deleted joins in [8]). Following [16], put $r = p^\alpha$ and consider the group $G = (\mathbb{Z}_p)^\alpha$ and let G act on the factors of the deleted join transitively. The rest of the reasoning is based on the following facts from [17, 18]:

Let X be a connected G-space. Consider the Leray–Serre spectral sequence with

$$E_2^{*,*} = H^*(BG; H^*(X; \mathbb{F}_p))$$

converging to $H^*_G(X; \mathbb{F}_p)$. Here G may act on $H^*(X; \mathbb{F}_p)$ so the cohomology $H^*(BG; \cdot)$ may be with twisted coefficients.

Definition 5.3. Denote by $i_G(X)$ the minimum r such that the differential d_r of this spectral sequence has nontrivial image in the bottom row.

The index i_G has the following properties, if G is a p-torus $G = (\mathbb{Z}_p)^\alpha$:

(1) (Monotonicity) If there is a G-map $f : X \to Y$, then $i_G(X) \le i_G(Y)$. If in addition $i_G(X) = i_G(Y) = n+1$ then the map $f^* : H^n(Y;\mathbb{F}_p) \to H^n(X;\mathbb{F}_p)$ is nontrivial.
(2) (Dimension upper bound) $i_G(X) \le \operatorname{hdim}_{\mathbb{F}_p} X + 1$.
(3) (Cohomology lower bound) If X is connected and acyclic over $\mathbb{F}_p$ in degrees $\le N-1$, then $i_G(X) \ge N+1$.

Now note that from the cohomology lower bound it follows that $i_G\left(K^{*r}_{\Delta(2)}\right) \ge m+1$, from the dimension upper bound it follows that $i_G\left(W^{*r}_{\Delta(r)}\right) \le m+1$, and from (1) the map

$$\tilde{f}^* : H^m\left(W^{*r}_{\Delta(r)};\mathbb{F}_p\right) \to H^m\left(K^{*r}_{\Delta(2)};\mathbb{F}_p\right)$$

must be nontrivial. From the cohomology exact sequence of a pair it follows that the natural map

$$g^* : H^m\left(W^{*r};\mathbb{F}_p\right) \to H^m\left(W^{*r}_{\Delta(r)};\mathbb{F}_p\right)$$

is surjective because $H^{m+1}(W^{*r}, W^{*r}_{\Delta(r)};\mathbb{F}_p) = 0$ by dimensional considerations. Now it follows that the map

$$(g \circ \tilde{f})^* : H^m\left(W^{*r};\mathbb{F}_p\right) \to H^m\left(K^{*r}_{\Delta(2)};\mathbb{F}_p\right)$$

is nontrivial. But the map $g \circ \tilde{f}$ is a composition of the natural inclusion

$$h : K^{*r}_{\Delta(2)} \to K^{*r}$$

with the map

$$f^{*r} : K^{*r} \to W^{*r}.$$

The latter map has contractible domain, and therefore induces a zero map on cohomology $H^m(\cdot;\mathbb{F}_p)$. We obtain a contradiction. $\square$

6. The case $r = 2$ of Theorem 1.1 and the Alexandrov width

Let us give a definition, generalizing the definition in [14]. The reader may also consult the book [10] in English. Throughout this section we use the notation

$$\delta A = \{\delta a : a \in A\} \quad \text{and} \quad A + B = \{a + b : a \in A,\ b \in B\}.$$

Definition 6.1. Let $K \subseteq \mathbb{R}^n$ be a convex body. Denote by $b_k(K)$ the maximal number such that for any map $K \to Y$ to a k-dimensional polyhedron there exists $y \in Y$ such that for any $\delta < b_k(K)$ the set $f^{-1}(y)$ cannot be covered by a translate of δK.

We use k-dimensional polyhedra Y following [14], but we may also use k-dimensional metric spaces as above.

The definition of the *Alexandrov width* (in [14]) is a bit different: A subset X of some normed space E is considered and $a_k(X)$ denotes the maximal number such that for any map $X \to Y$ to a k-dimensional polyhedron there exists $y \in Y$ such that for any $\delta < a_k(X)$ the set $f^{-1}(y)$ cannot be covered by a *ball* (in the given norm of E) of radius δ.

In [14, Theorem 1, p. 268] the results of K. Sitnikov and A.M. Abramov [1, 13] are cited, which assert that $a_k(X) = 1$ for any $k \le n-1$, if X is the unit ball of a norm in $\mathbb{R}^n$. In terms of Definition 6.1 this means that $b_k(K) = 1$ for centrally symmetric convex bodies in $\mathbb{R}^n$ if $k \le n-1$ and obviously $b_k(K) = 0$ for $k \ge n$.

Note that Theorem 1.1 for $r = 2$ actually asserts that $b_k(\Delta^n) = 1$ if $k \le n-1$. Indeed, if $f^{-1}(y)$ intersects all facets of Δ^n then it cannot be contained in a smaller homothetic copy of Δ^n. Now it makes sense to extend the result of K. Sitnikov and A.M. Abramov to (possibly not symmetric) convex bodies:

Theorem 6.2. *If K is a convex body in $\mathbb{R}^n$ and $k \le n-1$, then $b_k(K) = 1$.*

Proof. The proof in [14, Proposition 1, pp. 84–85] actually works in this case. Assume the contrary: the map $f : K \to Y$ is such that every preimage $f^{-1}(y)$ can be covered by a smaller copy of K and $\dim Y \le n-1$. For a fine enough finite closed covering of Y its pullback covering $\mathcal{U}$ of K has the following properties: the multiplicity of $\mathcal{U}$ is at most n and any $X \in \mathcal{U}$ can be covered by a translate of δK for some fixed $0 < \delta < 1$.

Assume $0 \in \operatorname{int} K$ and call the point t *the center* of a translate $\delta K + t$. Assign to any $X \in \mathcal{U}$ the center t_X of $\delta K + t_X \subseteq X$. Using the partition of unity subordinate to $\mathcal{U}$ we map K to the nerve of $\mathcal{U}$, and then map this nerve to at most $(n-1)$-dimensional subcomplex of $\mathbb{R}^n$ by assigning t_X to X. Finally we obtain a continuous map $\varphi : K \to \mathbb{R}^n$ such that for any $x \in K$ we have $x \in \varphi(x) + \delta K$ and the image $\varphi(K)$ has dimension $\le n-1$.

Under the above condition the image $\varphi(\partial K)$ cannot intersect εK if $\varepsilon < 1-\delta$, because $\varepsilon K + \delta K$ is in the interior of K. If we compose $\varphi|_{\partial K}$ with the central projection of $K \setminus \{0\}$ onto ∂K, we obtain a map homotopic to the identity map of ∂K. Therefore the map of pairs $\varphi : (K, \partial K) \to (K, K \setminus \varepsilon K)$ has degree 1, and $\varphi(K) \supseteq \varepsilon K$. Therefore $\varphi(K)$ is n-dimensional, which is a contradiction. □

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